AN INVESTIGATION OF THE CHUNG-FELLER THEOREM

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Abstract. In this paper, we shall prove the Chung-Feller Theorem in several ways. We provide an inductive proof, bijective proof, and proofs using generating functions, and the Cycle Lemma of Dvoretzky and Motzkin [2].

1. Introduction

The main focus of this paper is to prove the following result of Chung and Feller [1]:

The Chung Feller Theorem. *The number of paths from* $(0,0)$ *to* (n,n)*, with steps* $(0,1)$ *and* $(1,0)$ *and exactly* $2k$ *steps above the line* $x=y$ *is independent of* k*, for every* k *such that* $0 \le k \le n$*. In fact, it is equal to the* n*th Catalan number.*

2. Preliminary Definitions and an Inductive Proof

A Dyck path is a path with steps $(0,1)$ and $(1,0)$ that starts at the origin and ends at (n,n) for some positive integer n. We can study the number of paths indicated in the above Chung-Feller theorem in a simpler form if we generalize Dyck paths as seen in [3]. A *k-negative path* of length $2n$ is a path from $(0,0)$ to $(2n,0)$, with steps $(1,1)$ and $(1,-1)$, such that exactly $2k$ of these steps are below the horizontal axis.

A *prime Dyck path* is a Dyck path that returns to the x-axis only once at the end of the path. A *negative prime Dyck path* is a a prime Dyck path reflected about the x-axis. In general, any *negative* Dyck path is the reflection about the x-axis of a Dyck path.

An Inductive Proof of the Chung-Feller Theorem. As a base case, for $n=0$, there is only one path with 0 steps. So clearly there is only $C_0 = 1$ path with 0 steps below the x-axis.

Now, let $n > 0$ and assume that for all i such that $0 \le i < n$, the number of l-negative paths of length $2i$ is equal to C_i for all $l \le i$. Therefore, choose some $k \le n$.

Any nonempty k-negative path either starts out with either a prime Dyck path or a negative prime Dyck path. In the first case, the k-negative path starts with an up-step followed by a Dyck path of some length $2p-2$ which is then followed by a down-step and a k-negative path of length $2n-2p$, for some $p \le n-k$. On the other hand, the second case deals with paths that start out with a down-step, followed by a $(q-1)$-negative path of length $2q-2$ which is followed by an up-step and a $(k-q)$-negative path of length $2n-2q$, for some $q \le k$.

Let $\mathcal{N}_{\mathrm{up}}$ be the number of k-negative paths of length $2n$ that start with an up-step. Let $\mathcal{N}_{\mathrm{down}}$, similarly, be the number of k-negative paths of length $2n$ that start with a down-step. Let $\mathcal{N}$ be the total number of k-negative paths of length $2n$.

If a path starts out with an up-step and a Dyck path of length $2p-2$, then the number of such paths is the total number of Dyck paths of length $2p-2$ multiplied by the number

of k-negative paths of length $2n-2p<2n$. By the inductive hypothesis, this means that the number N_p^+ of such paths is $C_{p-1}C_{n-p}$. For the total number of k-negative paths that start off with an up-step we need to take the sum of N_p^+, for all $1\le p\le n-k$. Therefore, we have that the total number of k-negative paths of length $2n$ that start with an up-step is

$$\mathcal{N}_{\text{up}}=\sum_{p=1}^{n-k}N_p^+=\sum_{p=1}^{n-k}C_{p-1}C_{n-p}.$$

Similarly, we define N_q^- as the number of k-negative paths that start out with a down-step and a negative Dyck path of length $2q-2$. So N_q^- is equal to the number of negative Dyck paths of length $2q-2$ multiplied by the number of $(k-q)$-negative paths of length $2n-2q<2n$. Obviously, since the negative Dyck paths are reflections of (positive) Dyck paths, the number of negative Dyck paths of length $2q-2$ is equal to the number of Dyck paths of length $2q-2$. Therefore, we again have the sum of products of two Catalan numbers, and the total number of k-negative paths of length $2n$ that start with a down-step is

$$\mathcal{N}_{\text{down}}=\sum_{q=1}^{k}C_{q-1}C_{n-q}=\sum_{q=1}^{k}C_{n-q}C_{q-1}.$$

Therefore the total number of k-negative paths of length $2n$ is equal to

$$\mathcal{N}=\mathcal{N}_{\text{up}}+\mathcal{N}_{\text{down}}=\sum_{p=1}^{n-k}C_{p-1}C_{n-p}+\sum_{q=1}^{k}C_{n-q}C_{q-1}; \tag{1}$$

that is,

$$\mathcal{N}=(C_0C_{n-1}+C_1C_{n-2}+\cdots+C_{n-k-1}C_k)+(C_{n-1}C_0+C_{n-2}C_1+\cdots+C_{n-k}C_{k-1}).$$

Reversing the order of the summands in the second parentheses clearly gives

$$\begin{aligned}\mathcal{N}&=(C_0C_{n-1}+\cdots+C_{n-k-1}C_k)+(C_{n-k}C_{k-1}\cdots+C_{n-1}C_0)\\&=\sum_{i=0}^{n-1}C_iC_{n-i-1}.\end{aligned}$$

Now, for any Dyck path of length $2n$ we can look at the first nonempty prime path. It will have length $2i+2$ for some $i\ge 0$, giving a total of C_i such prime paths. Since the total length of the entire path is $2n$, we must have $i\le n-1$. Therefore, since the rest of the path is an arbitrary Dyck path of length $2n-2i-2$ there are C_{n-i-1} possible paths that can follow the initial prime path of length $2i+2$. Therefore, the total number of Dyck paths of length $2n$ that start with a prime path of length $2i+2$ is given by C_iC_{n-i-1}. Therefore if we sum over $0\le i\le n-1$ we will get all possible Dyck paths of length $2n$, which is given by C_n. Therefore

$$\mathcal{N}=\sum_{i=0}^{n-1}C_iC_{n-i-1}=C_n,$$

and so the total number of k-negative paths of length $2n$ is equal to C_n. $\square$

3. A Bijective Proof

Let S_k denote the set of all k-negative paths of length $2n$. Choose some k, such that $0 \leq k < n$.

For $s \in S_k$, find the last positive prime Dyck path q. Next factor s into $s = pqr$, where p is the path up to the last positive prime Dyck path and r is the rest of the path after q. Since q is the last positive prime Dyck path, the remainder of the path r must be a negative Dyck path. This factorization is unique.

Like any other prime Dyck path, q is composed of an up-step followed by an arbitrary Dyck path Q and a down-step. So q can be rewritten as $q = uQd$, where u denotes an up-step and d denotes a down-step. Define the function $\varphi_+ : S_k \to S_{k+1}$ by

$$\varphi_+(s) = \varphi_+(puQdr) = pdruQ, \quad \text{for any} s \in S_k. \tag{2}$$

Given an $s \in S_k$, $\varphi_+(s) = pdruQ$ by (2). Since p begins and ends on the the x-axis, the step d is below the x-axis. As noted earlier r is an arbitrary negative Dyck path, so in the path $\varphi_+(s)$, r starts and ends at height -1, without going above it. Therefore the step u is negative, starting at height -1 and ends on the x-axis. Since Q is a Dyck path it contains no negative steps, so altogether $\varphi_+(s)$ has 2 steps below the x-axis in addition to the number of negative steps in path segments p and r. Since the only negative steps in the path $s \in S_k$ occur in p and r, the number of negative steps in path segments p and r is equal to $2k$. Thus $\varphi_+(s)$ has $2k+2$ steps below the x-axis and so is a member of S_{k+1}.

Now for any $\sigma \in S_{k+1}$ we can find the last negative prime path ω. Thus we can write $\sigma = \pi\omega\varrho$, where π is the path up to the last negative prime, and ϱ is the rest of the path which is by construction positive. So since ω is a negative prime path it can be written as $d\Omega u$, where Ω is an arbitrary negative Dyck path. Therefore any $\sigma \in S_{k+1}$ can be uniquely written as $\sigma = \pi d\Omega u\varrho$, where Ω is a negative path and ϱ is a positive path. Therefore, if $s_1 = p_1uQ_1dr_1 \neq s_2 = p_2uQ_2dr_2 \in S_k$ then either $p_1 \neq p_2$, $Q_1 \neq Q_2$ or $r_1 \neq r_2$, so clearly since $\varphi_+(s_1) = p_1dr_1uQ_1, \varphi_+(s_2) = p_2dr_2uQ_2$, since the factorization in S_{k+1} is unique, $\varphi_+(s_1) \neq \varphi_+(s_2)$. Therefore, φ_+ is injective.

Using the factorization on S_{k+1} we can define the injection $\varphi_- : S_{k+1} \to S_k$ which sends $\sigma = \pi d\Omega u\varrho \mapsto \pi u\varrho d\Omega$. Since there are now two fewer negative steps in $\varphi_-(\sigma)$ than in σ itself, due to the fact that the up-step now comes before the down-step, $\varphi_-(\sigma) \in S_k$. Additionally, since there is an unique decomposition in S_k, φ_- is injective. Thus, by the Scröder-Bernstein theorem $|S_k| = |S_{k+1}|$. In fact, $\varphi_- = \varphi_+^{-1}$, since $\varphi_-(pdruQ) = puQdr$ for all $s = puQdr \in S_k$, so φ_+ is in fact a bijection.

Therefore, we can now bijectively prove the Chung-Feller theorem.

Bijective Proof. The cardinality of S_k and S_{k+1} are equal, for all nonnegative $k \leq n-1$. Since there are $n+1$ equal sets of paths of length $2n$ each with the same cardinality, and the total number of paths of length $2n$ is $\binom{2n}{n}$, we have the equation

$$(n+1)\mathcal{K} = \binom{2n}{n},$$

where $\mathcal{K}$ is the number of k-negative paths of length $2n$ for any k such that $0 \leq k \leq n$.

Therefore $\mathcal{K} = \frac{1}{n+1}\binom{2n}{n} = C_n$. □

4. A Generating Function Approach to the Theorem

In general a generating function is a very useful tool used in enumerative combinatorics. The generating function of an infinite sequence $\{a_n\}$ can be thought of as the function for which the coefficient of x^n in the power series expansion about $x = 0$, is a_n. That is, if $A(x)$ is the generating function for the sequence $\{a_n\}$, then $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

The generating function can also be obtained more formally by weighting certain combinatorial objects by a variable x. For instance, if $\{a_n\}$ represents the number of *bloops* with n *glurps*, then $A(x)$ is obtained by weighting each glurp by x and summing over all bloops. We will use both formulations in this section to give a straight-forward proof of the Chung-Feller theorem.

The generating function for the Catalan numbers is given by

$$c(x) = \sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}, \tag{3}$$

which is obtained by weighting every step in a Dyck path by $\sqrt{x}$ and summing over all possible Dyck paths.

Let us construct a generating function for arbitrary paths that end on the x-axis by weighting each step by $\sqrt{x}$ and each step below the x-axis by $\sqrt{tx}$, such that a path with $2n$ total steps, $2k$ of which are below the x-axis is given the weight $t^k x^n$. As described in section 2, the primes of a k-negative path are either positive prime Dyck paths or negative prime Dyck paths. Let P_n denote the number of (positive) prime Dyck paths of length $2n$.

Such a path of length $2n$ consists of an arbitrary Dyck path of length $2n-2$ sandwiched between an up-step and down-step. So the number of prime Dyck paths of length $2n$ is given by the $(n-1)$th Catalan number, that is $P_n = C_{n-1}$. Therefore the generating function for the prime Dyck paths is given by

$$p_+(x) = \sum_{n=1}^{\infty} P_n x^n = \sum_{n=1}^{\infty} C_{n-1} x^n = x \sum_{n=0}^{\infty} C_n x^n = xc(x). \tag{4}$$

Similarly, for negative prime Dyck paths the generating function, now weighted by $\sqrt{xt}$ because each step in a negative prime Dyck path is below the x-axis is given by

$$p_-(x,t) = \sum_{n=1}^{\infty} P_n (tx)^n = tx \sum_{n=0}^{\infty} C_n (tx)^n = txc(tx). \tag{5}$$

So since arbitrary paths can be factored into l primes (either positive or negative) for some $l \geq 0$, the generating function for such paths is

$$N(x,t) = \sum_{l=0}^{\infty} (p_- + p_+)^l = \frac{1}{1-(p_- + p_+)}.$$

In this generating function the coefficient of $t^i x^j$ is the number paths of total length $2j$ and with $2i$ steps below the x-axis.

By the definition of $c(x)$ and equations (4) and (5) we get that

$$\begin{aligned} N(t,x) &= \frac{1}{1-xc(x)-txc(tx)} \\ &= \frac{1}{1-(x\frac{1-\sqrt{1-4x}}{2x}+tx\frac{1-\sqrt{1-4tx}}{2tx})} \\ &= \frac{2}{\sqrt{1-4x}+\sqrt{1-4tx}}. \end{aligned}$$

Rationalizing the denominator we see that

$$\begin{aligned} N(t,x) &= \frac{\sqrt{1-4x}-\sqrt{1-4tx}}{2x(t-1)} \qquad (6) \\ &= \frac{1-\sqrt{1-4x}-1+\sqrt{1-4xt}}{2x(1-t)} \\ &= \frac{1}{1-t}\cdot\left(\frac{1-\sqrt{1-4x}}{2x}-\frac{1-\sqrt{1-4xt}}{2x}\right) \\ &= \frac{1}{1-t}\left(c(x)-tc(tx)\right) \\ &= \frac{1}{1-t}\left(\sum_{n=0}^{\infty}C_nx^n-\sum_{n=0}^{\infty}C_nt^{n+1}x^n\right) \\ &= \sum_{n=0}^{\infty}\frac{1-t^{n+1}}{1-t}C_nx^n. \end{aligned}$$

Using $N(t,x)$ and a simple algebraic identity we can prove the Chung-Feller Theorem.

Proof. Because

$$\frac{1-t^{n+1}}{1-t}=1+t+t^2+\cdots+t^n,$$

equation (6) can be rewritten as

$$N(t,x)=\sum_{n=0}^{\infty}C_n\left(\sum_{k=0}^{n}t^k\right)x^n. \qquad (7)$$

Thus the coefficient of t^kx^n in $N(t,x)$ is equal to C_n for $0\le k\le n$. Therefore the number of k-negative paths of length $2n$ is equal to C_n for $0\le k\le n$. $\square$

5. A Proof by Reordering

The Cycle Lemma of Dvoretzky and Motzkin [2] is intricately linked with our main theorem. In [4], the author provides the following proof of the Cycle Lemma using an ordering which naturally reveals the equidistribution of the Chung-Feller Theorem.

Lemma 1 (Cycle Lemma)**.** *Given a sequence $\pi=a_1a_2a_3\cdots a_n$ of integers with $a_i\le 1$ such that $\sum_{j=1}^{n}a_i=k>0$, there are exactly k values of i such that all partial sums of the cyclic permutation $\pi_i=a_{i+1}a_{i+2}\cdots a_{2n+k}a_1\cdots a_i$ are positive.*

This is a stronger claim than we need, so let us just restrict ourselves to $a_i \in \{-1, 1\}$. For ease of computation for any $p \leq n$, let $s(p) = \sum_{j=1}^{p} a_i$. Given a sequence π as above, let us define a new order relation $\triangleleft$ on $\{0, 1, \ldots, n\}$. For any $p, q \in \{0, 1, \ldots, n\}$,

$$p \triangleleft q,$$

if $s(p) < s(q)$ or if $s(p) = s(q)$ and $p > q$.

Additionally, let us define m_i for $i = 0, 1, \ldots, n$ such that $m_i \in \{0, 1, \ldots, n\}$ and there are exactly i elements $m \in \{0, 1, \ldots, n\}$ such that $m \triangleleft m_i$.

Now, let if we look at the jth cyclic shift of π, denoted $\pi_j = a_{j+1}a_{j+2}\cdots a_n a_1 \cdots a_j$, then the partial sums of π_j, $s^j(p)$, is given by $s^j(p) = \sum_{i=1}^{p-j} a_{i+j} = a_{i+1} + a_{i+2} + \cdots + a_p$, where indices are considered modulo n. This leads to the equation

$$s^j(p) = \begin{cases} s(p) - s(j), & \text{if } j \leq p \leq n; \\ s(p) - s(j) + k, & \text{if } 0 \leq p < j. \end{cases} \tag{8}$$

Proposition. *For any sequence π with steps $a_i \in \{-1, 1\}$ and sum $k = 1$, the m_ith cyclic shift of π has exactly $i + 1$ values of p such that $s^{m_i}(p) \leq 0$.*

Proof. Clearly, $s^{m_i}(m_i) = 0$, so there is at least one such value of p. If $l < i$, then let us check $s^{m_i}(m_l)$. If $s(m_l) = s(m_i)$ then $m_l > m_i$, so we know that $s^{m_i}(m_l) = s(m_l) - s(m_i) = 0$. If, however, $s(m_l) < s(m_i)$ then $s^{m_i}(m_l) \leq s(m_l) + 1 - s(m_i) < 1$, so $s^{m_i}(m_l) \leq 0$.

If $l > i$, then either $s(m_l) > s(m_i)$ or $m_l < m_i$ and $s(m_l) = s(m_i)$. In the first case since $s^{m_i}(m_l) \geq s(m_l) - s(m_i)$, clearly $s^{m_i}(m_l) > 0$. Otherwise, if $m_l < m_i$ then $s^{m_i}(m_l) = 1 + s(m_l) - s(m_i)$ so since $s(m_l) = s(m_i)$, $s^{m_i}(m_l) = 1$.

Thus there are exactly $i+1$ values of p, namely $m_0, m_1, \ldots, m_i$, such that $s^{m_i}(p) \leq 0$. □

The proof of the Cycle Lemma naturally follows from this proposition.

Proof of Cycle Lemma. Use the order relation $\triangleleft$ to calculate m_i for each $i = 0, 1, \ldots, n$ and let $\phi = \pi_{m_0}$. By definition, only $s^{m_0}(m_0) = 0$. Since $s^{m_0}(p) = s_\phi(p - m_0)$, only $s_\phi(0) \leq 0$, so all partial sums of ϕ for $p \geq 0$ are positive. □

Let π represent a Dyck path from $(0, 0)$ to $(2n + 1, 1)$, by taking the path $(1, a_i)$. Let us restrict the ordering $\triangleleft$ to just the cyclic shifts of π that begin with an up-step. Since the order structure of $\{0, 1, \ldots, 2n + 1\}$ is maintained we can order the up-steps $j_k = m_i$ is the up-step with exactly k up-steps j such that $j \triangleleft j_k$ for $k = 0, 1, 2, \ldots, n$. Therefore, though there will be a total of i vertices on or below the x-axis in the n_kth cyclic shift of π, there will be precisely k up-steps that start on or below the x-axis.

Since there are $n + 1$ total up-steps, there are $n + 1$ cyclic shifts of π that start with an up-step. Now since there is precisely one cyclic shift for every path, then we have that the total number of paths with steps $a_i \in \{-1, 1\}$ and sum 1 that start with an up-step and have exactly k up-steps that start on or below the x-axis is given by the fraction $\frac{1}{n+1}$ of the total number of paths from $(0, 0)$ to $(2n + 1, 1)$ that start with an up-step. Thus it is the same as the number of paths from $(1, 1)$ to $(2n+1, 1)$ which is precisely the number of paths from $(0, 0)$ to $(2n, 0)$; that is, $\binom{2n}{n}$.

Therefore the number of paths from $(0, 0)$ to $(2n + 1, 1)$ that start with an up-step and have exactly k up-steps that start on or below the x-axis, for $k = 0, 1, \ldots, n$, is given by

$\frac{1}{n+1}\binom{2n}{n} = C_n$. If we drop the initial up-step then we are left with a path from $(0,0)$ to $(2n,0)$ with exactly k up-steps below the x-axis. So there are C_n paths from $(0,0)$ to $(2n,0)$ with exactly k up-steps (and thus a total of $2k$ steps) below the x-axis for each k.

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